Prof. Hadley

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VOL. III. NO. III.

THE MATHEMATICAL MONTHLY.

DECEMBER, 1860.

EDITED BY

J. D. RUNKLE, A.M., A.A.S.

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1860.

Editorial Items.

Prizes for Vol. III. offered to all Students.

FOR SOLUTIONS. — A 1st Prize, $7.50; 2d, $ 5; 3d, $ 3; 4th, Vol. III. The third and fourth prizes are confined to beginners for solutions of Problems I. and II.; but as the best solutions only will be published, advanced students are desired to send solutions of all the problems.

FOR ESSAYS. — A 1st Prize, $ 50; 2d, $ 40; 3d, $ 30; 4th, $ 20; 5th, $ 10. The first and second of these prizes are open to all competitors. Instead of cash prizes, we propose to make them all payable in mathematical books. This change will make the prizes less burdensome to us while it will be appropriate, and we believe quite as acceptable to competitors, to receive standard mathematical books, — such as BOWDITCH's edition of *Mécanique Céleste*, PEIRCE's *Analytic Mechanics*, BARTLETT's *Analytic Mechanics*, DAVIS's edition of *Theoria Motus*, DAVIES and PECK's *Mathematical Dictionary*, CHAUVENET's *Analytic Trigonometry*. But selections may be made from SEVER & FRANCIS's list, and desired works not found there will be ordered. To those preferring standard works in other departments, we offer WORCESTER's *Quarto Dictionary* and the fine editions of HUME, GIBBON, LAMB, and MACAULAY, HAMILTON's *Metaphysics* and *Logic*, and APPLETON's *New American Encyclopædia*, described in our advertising pages. To this list we may hereafter add editions of other standard authors, if desired.

A Generous Offer.

It gives us great pleasure to announce that "A Friend of the Mathematical Monthly Prizes," whose name we are not permitted to give, wishes to assume the responsibility of the Prizes for the solutions of the Problems in the number for December. He says, "The Monthly has been of great value to me. It has been my tutor, and certainly the tuition is cheap. I ought to be able now to contribute something more to its sustenance." The solutions will be due February 1, 1861, and the Report will be published in the March number. We hope to receive a large number of solutions as evidence of an exciting contest for these Prizes. It will be seen (Editorial Items), that the times for the solutions of the September and October Problems are extended one month. Hereafter we shall issue the Monthly promptly on or before the first of the month.

Portrait of Sir John Herschel.

Through the kindness of MISS MARIA MITCHELL we have received a fine photograph of SIR JOHN HERSCHEL, which we shall have engraved on steel for an early number.

We expect our friends will aid us by increasing our subscription list. See Mathematical Monthly, on seventh page of Advertiser.

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THE TRIBUNE FOR 1861.

PROSPECTUS.

The XXth Volume of THE WEEKLY TBIBUNE commenced with the issue of September 1. During the past year THE TRIBUNE has been obliged to devote quite a large proportion of its space to Politics, but we shall soon be able to forego political discussion almost entirely, for months if not for years, and devote nearly all our columns to subjects of less intense, but more abiding, interest. Among these, we mean to pay especial attention to

I. EDUCATION. — The whole subject of Education, both Popular and General, will be discussed in our columns throughout the year 1861, and we hope to enlist in that discussion some of the profoundest thinkers and the ablest instructors in our country. It is at once our hope and our resolve that the cause of Education shall receive an impetus from the exertions of THE TRIBUNE in its behalf during the year 1861.

II. AGRICULTURE. — We have been compelled to restrict our elucidations of this great interest throughout 1860, and shall endeavor to atone therefor in 1861. Whatever discovery, deduction, demonstration, is calculated to render the reward of labor devoted to cultivation more ample or more certain, shall receive prompt and full attention.

III. MANUFACTURES, &c. — We hail every invention or enterprise whereby American Capital and Labor are attracted to and advantageously employed in any department of Manufacturing or Mechanical Industry as a real contribution to the Public Weal, insuring ampler, steadier, more convenient, more remunerating markets to the Farmer, with fuller employment and better wages to the Laborer. The Progress of Mining, Iron-making, Steel-making, Cloth-weaving, &c., &c., in our country, and the world, shall be watched and reported by us with an earnest and active sympathy.

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V. HOME NEWS. — We employ regular paid correspondents in California, at the Isthmus of Darien, in the Rocky Mountain Gold Region, and wherever else they seem requisite. From the more accessible portions of our own country, we derive our information mainly from the multifarious correspondents of the Associated Press, from our exchanges, and the occasional letters of intelligent friends. We aim to print the cheapest general newspaper, with the fullest and most authentic summary of useful intelligence, that is anywhere afforded. Hoping to "make each day a critic on the last," and print a better and better paper from year to year, as our means are steadily enlarged through the generous co-operation of our many well-wishers, we solicit and shall labor to deserve a continuance of public favor.

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Agents will have two months between this time and the first of February, 1861, to canvass their respective towns, and may send in names as they obtain them. The names of the persons receiving the county premiums, with their towns and the number on their respective lists, will be published in the February number.

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THE

MATHEMATICAL MONTHLY.

Vol. III. . . . DECEMBER, 1860. . . . No. III.

PRIZE PROBLEMS FOR STUDENTS.

I. From AC, the diagonal of a square $ABCD$, cut off AE equal to one fourth of AC, and join BE, DE. Show that the figure $BADE$ equals twice the square on AE.

II. Two toothed wheels work against each other. Show that, if the number of teeth in one be prime to that in the other, before two teeth, which have once been in contact, come in contact again, every tooth of the one wheel will have been in contact with every tooth of the other.

III. Eliminate φ from the equations

$$x \cos (\varphi + \alpha) + y \sin (\varphi + \alpha) = a \sin 2\varphi,$$
$$y \cos (\varphi + \alpha) + x \sin (\varphi + \alpha) = 2a \cos 2\varphi,$$

showing that

$$(x \sin \alpha - y \cos \alpha)^{\frac{2}{3}} + (y \sin \alpha + x \cos \alpha)^{\frac{2}{3}} = (2a)^{\frac{2}{3}}.$$

IV. Show that $\frac{1}{4} m^2$ is the area of the greatest triangle which can be formed with the lines a, b, c, subject to the condition $a^3 + b^3 + c^3 = 3m^3$.

V. The equation of a family of ellipses is $ax^2 + by^2 = 1$, when $a - b = c$, a constant. Show that the curve, which cuts all the individuals at an angle whose tangent is $\frac{y}{x}$, is $x = Ce^{-\frac{cy^2}{2}}$.

Solutions of these problems must be received by February 1, 1861.

NOTES AND QUERIES.

1. *Perfect Squares.* — It is a well-known fact that every perfect square, when represented by the Arabic or Hindoo method, must end in 0, 1, 4, 5, 6, or 9. This fact enables us to decide, at a glance, that all numbers ending in 2, 3, 7, or 8 are not perfect squares; but it does not aid us in deciding which of the numbers ending in 0, 1, 4, 5, 6, or 9 are perfect squares, for the converse of the proposition stated above is not true. To assist, to some extent, in removing this difficulty, I submit the following discussion.

PROPOSITION. — The last two figures of the square of any integer must correspond to the last two figures of the square of some number in the series 1, 2, . . 99. The endings of the squares of the first twenty-five numbers form the series 01, 04, 09, 16, 25, 36, 49, 64, 81, 00, 21, 44, 69, 96, 25, 51, 89, 24, 61, 00, 41, 84, 29, 71, 24. This series also represents the endings of the squares of the numbers from 51 to 75, inclusive. The series, taken backwards, represents the endings of the numbers from 25 to 49, and from 75 to 99, inclusive. This shows that the squares of the numbers equally above and below 25 and 75 end in the same two figures. Omitting the like endings in the above series, and arranging in order, we have 00, 01, 04, 09, 16, 21, 24, 25, 29, 36, 41, 44, 49, 56, 61, 64, 69, 76, 81, 84, 89, 96. These endings indicate the following facts: Every perfect square ending in 1, 4, or 9 must have before the last figure one of the even numbers 0, 2, 4, 6, or 8; ending in 6, must be preceded by one of the odd numbers 1, 3, 5, 7, or 9; ending in 5, by 2; and ending in 0, by 0. — Prof. W. D. HENKLE, Normal School, Lebanon, Ohio.

2. *Problem.* — Prove that, when x is a whole number and greater than one, $x^3 - x$ is divisible by 6.

Suppose that $x^3 - x$ is divisible by 6 when $x = p$, then will it be divisible by 6 when $x = p + 1$. For, letting $x = p + 1$, we have

$$x^3 - x = p^3 + 3p^2 + 3p + 1 - (p+1) = p^3 + 3p^2 + 2p = p^3 - p + 3p^2 + 3p.$$

Now $p^3 - p$ is divisible by 6, by hypothesis; and if we can prove that $3p^2 + 3p$ is also divisible by 6, then it will follow that $x^3 - x$ is divisible by 6 when $x = p + 1$. But to prove that $3p^2 + 3p$ is divisible by 6, we must prove that $p^2 + p$ is divisible by 2. Suppose that $p^2 + p$ is divisible by 2 when $p = r$, then will it be divisible by 2 when $p = r + 1$. For, putting $p = r + 1$, we get

$$p^2 + p = r^2 + 2r + 1 + r + 1 = r^2 + 3r + 2 = r^2 + r + 2r + 2.$$

But $r^2 + r$ is divisible by 2 by hypothesis, and $2r + 2$ is also divisible by 2. Now we know that $p^2 + p$ is divisible by 2 when $p = 1$, $\therefore$ it must be when $p = 2, 3, 4$, &c.; and therefore if $x^3 - x$ is divisible by 6 when $x = p$, it will be when $x = p + 1$. But we know by inspection that it is divisible when $x = 2$; $\therefore$ it will be when $x = 3, 4, 5$, &c. — John Scott, Jr., Night High School, Cincinnati, Ohio.

3. *Note on the Prize Solution of Problem III.*, p. 25. — As there has been a slight typographical error each time the formulas have been printed, I send them in their correct and most reduced form. As m and n exchange values in the two solids, the formulas take very nearly the same forms, one having cosine where the other has the square of the cotangent, and 3 in the place of 2. The last forms given below are adapted to logarithmic computation.

$$\text{Vol. Dodec.} = \frac{5e^3 \cot^2 \beta}{2\sqrt{(4\sin^2\beta - 1)}} = e^3 \left(\frac{15 + 7\sqrt{5}}{4}\right) = \frac{5e^3 \tan^2 54^\circ}{4\sqrt{(\sin 6^\circ \cos 24^\circ)}}.$$

$$\text{Vol. Icos.} = \frac{5e^3 \cos \beta}{3\sqrt{(3 - 4\cos^2\beta)}} = \frac{5e^3 \cos\beta}{3\sqrt{(4\sin^2\beta - 1)}}$$

$$= e^3 \left(\frac{15 + 5\sqrt{5}}{12}\right) = \frac{5e^3 \sin 54^\circ}{6\sqrt{(\sin 6^\circ \cos 24^\circ)}}.$$

— Prof. D. W. Hoyt, New Hampton Institution, Fairfax, Vt.

4. *Correction of the Method of Computing Six per Cent Interest for Days.* — The usual method of computing interest at six per cent, for a given

number of days, that is, "Divide the principal by 6,000 and multiply the quotient by the given number of days," is not of course accurate when large numbers are involved; but the following development, p, denoting the principal, and d the days,

$$\frac{0.06\,p\,d}{365.25} = \frac{p\,d}{6000}\left\{1 + \tfrac{1}{70}\left[1 + \tfrac{1}{50}\left(1 + \tfrac{1}{24}\right)\right]\right\},$$

shows that the correction is almost exactly $\frac{1}{70}$ of the first result; and if this is not sufficient, the second correction is $\frac{1}{50}$ of the first. The third correction, $\frac{1}{24}$ of the second, will never be needed, and is not quite so convenient to apply mentally as the others, but it is accurate to 0.000001 of its value. It will be noticed that these corrections are all additive. All except the first, however, are more curious than useful. — B.

5. *Solution of Problem V., No. IX., Vol. II.* — Two great circles are drawn at random on a sphere. What is the probability that their mutual inclination, taken less than 90°, will be contained between any given limits, as $n°$ and $m°$?

1st. The mutual inclination of the two circles will be equal to the angular distance of their poles.

2d. The pole of either great circle is just as likely to fall in one position as in any other; hence the probability that it will fall in any given portion of the surface of the sphere is proportional to the superficial area of that portion.

3d. In order that the angular distance of the poles may be between the limits $n°$ and $m°$, one pole must fall within the zone contained between two small circles at the respective distances of $n°$ and $m°$ from the other pole.

4th. Hence the probability required will be equal to the area of this zone divided by the area of the hemisphere, or to $\cos n° - \cos m°$. LAPLACE, in considering the inclinations of the orbits of the planets

and comets, in the *Théorie Analytique des Probabilités,* fell into the error of supposing that this probability would be equal to $\frac{m^\circ - n^\circ}{90^\circ}$; or that all inclinations would be equally probable. — S. Newcomb, Cambridge, Mass.

6. *Note on the Powers of Binomials.* — Find the fifth power of the binomial $\frac{1}{2}x - 3y^2$.

1	$+5$	$+10$	$+10$	$+5$	$+1,$	Coefficients.
$\frac{1}{32}x^5$	$+\frac{1}{16}x^4$	$+\frac{1}{8}x^3$	$+\frac{1}{4}x^2$	$+\frac{1}{2}x$	$+1,$	Power of $\frac{1}{2}x$.
1	$-3y^2$	$+9y^4$	$-27y^6$	$+81y^8$	$-243y^{10},$	Powers of $-3y^2$.

$$\tfrac{1}{32}x^5 - \tfrac{15}{16}x^4y^2 + \tfrac{45}{4}x^3y^4 - \tfrac{135}{2}x^2y^6 + \tfrac{405}{2}xy^8 - 243y^{10} = (\tfrac{1}{2}x - 3y^2)^2.$$

This simple method of writing powers of complicated binomials is sufficiently obvious. Writing the different lines, and then multiplying together the terms which stand in the same column, so divides up the process, that the whole is made plain and easy to the student. — Davies's University Algebra.

7. *The arithmetic mean of any number of positive quantities is greater than the geometric mean*; that is,

$$\frac{a + b + c + d \ldots + k}{n} > (a\,b\,c\,d \ldots k)^{\frac{1}{n}}.$$

Let P denote $(a\,b\,c\,d \ldots k)^{\frac{1}{n}}$, and Q denote $\frac{a+b+c+d\ldots k}{n}$. Suppose a and b respectively the greatest and least of the n quantities $a, b, c, d, \ldots k$; and let $a_1 = b_1 = \frac{1}{2}(a+b)$, and let $P_1 = (a_1 b_1 c d \ldots k)^{\frac{1}{n}}$; then, since $a_1\,b_1 > a\,b$, we have $P_1 > P$. Next, if the factors in P_1 be not all equal, remove the greatest and least of them, and put in their places two new factors, each equal to half the sum of those removed; let P_2 denote the new geometrical mean; then $P_2 > P_1$. If we proceed in this way, we obtain a series, $P, P_1, P_2, P_3, \ldots P_r$, each term of which is greater than the preceding term; and by

taking r large enough, we may have the factors of P_r as nearly equal as we please; thus, when r is large enough, we may consider $P_r = Q$, therefore P is less than Q. — TODHUNTER's Algebra.

8. *Note on the Successive Derivatives of tan φ.* — We have

$$D \tan \varphi = D \frac{\sin \varphi}{\cos \varphi} = \frac{\cos^2 \varphi + \sin^2 \varphi}{\cos^2 \varphi} = \frac{1}{\cos \varphi^2}.$$

$$D^2 \tan \varphi = -\frac{2\, D \cos \varphi}{\cos^3 \varphi} = \frac{2 \sin \varphi}{\cos^3 \varphi} = \frac{2 \tan \varphi}{\cos^2 \varphi} = 2 \tan \varphi \; D \tan \varphi.$$

For brevity, let us adopt the notation $\tan \varphi = D^0$, $D \tan \varphi = D^1$, $D^2 \tan \varphi = D^2$, &c. Then

$$D^2 \tan \varphi = 2\, D^1 D^0,$$

$$D^3 \tan \varphi = 2\, D^2 D^0 + 2 (D^1)^2,$$

$$D^4 \tan \varphi = 2\, D^3 D^0 + 2\, D^2 D^1 + 4\, D^2 D^1 = 2\, D^3 D^0 + 6\, D^2 D^1,$$

$$\begin{aligned} D^5 \tan \varphi &= 2\, D^4 D^0 + 2\, D^3 D^1 + 6\, D^3 D^1 + 6 (D^2)^2 \\ &= 2\, D^4 D^0 + 8\, D^3 D^1 + 6 (D^2)^2, \end{aligned}$$

$$\begin{aligned} D^6 \tan \varphi &= 2\, D^5 D^0 + 2\, D^4 D^1 + 8\, D^4 D^1 + 8\, D^3 D^2 + 12\, D^3 D^2 \\ &= 2\, D^5 D^0 + 10\, D^4 D^1 + 20\, D^3 D^2. \end{aligned}$$

In general,

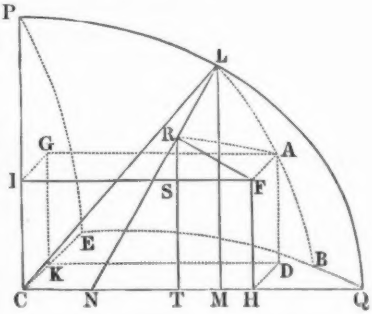
$$\begin{aligned} D^n \tan \varphi &= (D^1 + D^0)^{n-1} \\ &= D^{n-1} D^0 + (n-1)\, D^{n-2} D^1 + \frac{(n-1)(n-2)}{1.2} D^{n-3} D^2 + \frac{(n-1)(n-2)(n-3)}{1.2.3} D^{n-4} D^3 + \&c. \end{aligned}$$

In this series, terms equally distant from the extremes are equal, and may be added together. These very curious and interesting forms are found in PEIRCE's *Curves and Functions*, Vol. I.

THE EARTH CONSIDERED AS A SPHEROID OF REVOLUTION.—GEODETIC FORMULAS.

By Dr. R. C. Matthewson, U. S. Deputy Surveyor, San Francisco.

Let $CEPQ$ represent one eighth of the spheroid, bounded by the convex surface EPQ, the plane of the equator EQC, and any other two planes, CPE and CPQ, at right angles to each other, passing through the pole P, and the centre C. On the plane CPQ,* from any point L in the meridian PQ, draw the radius LC, the normal LN, and the ordinate LM. Let the meridional plane CPQ be intersected in L by the normal plane LNB, making with it any angle BLQ, BL being the intersection with the surface. Then the curve BL, as well as PQ, will be the arc of an ellipse. In BL take any point A, and from it let fall the perpendiculars AD, AF, and AG respectively upon the planes CQE, CPE, and CPQ, and complete the parallelograms AH, AI, HK, and IK. On the plane CPQ draw FR at right angles to LN, and RT at right angles to CQ, intersecting FI in S, and join AR.



Put $CM = x$ and $LM = y$, the rectangular co-ordinates of the point L of the meridional ellipse PQ, C being the origin; $NR = x'$ and $AR = y'$, the rectangular co-ordinates of the point A of the normal ellipse BL, N being the origin; $AD = x''$, $AF = y''$, and $AG = z''$, the rectangular co-ordinates of the point A of the spheroidal surface EPQ, C being the origin; $CQ = a$, the equatorial

* For sake of distinction, the lines upon the plane CPQ are continuous, while those not upon it are dotted.

radius; $CP = b = a(1-e^2)^{\frac{1}{2}}$, the semi-axis, e being the eccentricity; $LN = n$, the normal; $MN = s$, the subnormal; $CN = m$, the adnormal; $CL = r$, the radius of the spheroid at the point L; the angle $LNM = l$, the geographical latitude of the point L; the angle $LCM = l'$, the geocentric latitude of the same point and the angle $ALQ = z$, the azimuth of the point A from L.

Then the equation of the meridional ellipse PQ is

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

which, by eliminating b, becomes $y^2 = (1-e^2)(a^2 - x^2)$. Differentiating this equation, we find the length of the subnormal,

$$s = y\frac{dy}{dx} = (1-e^2)\,x = n\cos l.$$

Hence $\qquad x = \frac{n\cos l}{1-e^2}$, and $y = n\sin l$.

Substituting these values for x and y in the equation of the meridional ellipse, we have the normal, $n = \frac{a(1-e^2)}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$; and consequently the abscissa, $x = \frac{a\cos l}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$; the ordinate, $y = \frac{a(1-e^2)\sin l}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$;

the subnormal, $\quad s = (1-e^2)\,x = \frac{a(1-e^2)\cos l}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$;

the adnormal, $m = x - (1-e^2)\,x = e^2 x = \frac{a e^2\cos l}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$;

the radius, $\quad r = (x^2+y^2)^{\frac{1}{2}} = a\left(1 - \frac{e^2(1-e^2)\sin^2 l}{1-e^2\sin^2 l}\right)^{\frac{1}{2}}$;

and the geocentric latitude,

$$l' = \frac{s}{x}\tan l = \frac{(1-e^2)\,x}{x}\tan l = (1-e^2)\tan l.$$

Again, the equation of the spheroidal surface EPQ is

$$a^2 x'^2 + b^2 y'^2 + b^2 z'^2 = a^2 b^2,$$

or, by eliminating b, $\quad x'^2 + (1-e^2)(y'^2 + z'^2 - a^2) = 0$. Now

in the right-angled triangle AFR we have the hypotenuse $AR = y'$, and the angle $ARF = z$, whence $AF = y' \sin z$ and $FR = y' \cos z$; in the right-angled triangle FRS we have the hypotenuse $FR = y' \cos z$ and the angle $FRS = l$, whence $FS = y' \sin l \cos z$ and $RS = y' \cos l \cos z$; and in the right-angled triangle NRT we have the hypotenuse $NR = x'$ and the angle $RNT = l$, whence $RT = x' \sin l$ and $NT = x' \cos l$. Hence,

$$x'' = AD = IC = RT - RS = x' \sin l - y' \cos l \cos z;$$

$$y'' = AF = CK = y' \sin z;$$

$$z'' = AG = CH = FS + NT + CN = y' \sin l \cos z + x' \cos l + m.$$

Now if we substitute the foregoing values for x'', y'', and z'' in the equation of the spheroidal surface EPQ, we shall find the equation of the normal ellipse BL to be

$$A x'^2 + B y'^2 + C x' y' + D x' + E y' + F = 0,$$

in which $A = 1 - e^2 \cos^2 l$; $B = 1 - e^2 + e^2 \cos^2 l \cos^2 z$;

$$C = -2 e^2 \sin l \cos l \cos z; \qquad D = \frac{2 a e^2 (1 - e^2) \cos^2 l}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}};$$

$$E = \frac{2 a e^2 (1 - e^2) \sin l \cos l \cos z}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}}; \text{ and } F = -(1 - e^2)(a^2 - m^2).$$

Differentiating the equation of the normal ellipse BL twice, and, in each of the differential coefficients, making $x' = n = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}}$ and $y' = 0$, the values of the co-ordinates when the point A coincides with L, we have

$$\frac{dx'}{dy'} = 0, \qquad \text{and} \qquad \frac{d^2 x'}{dy'^2} = -\frac{2D}{2Ax' + D}.$$

Substituting these values for $\frac{dx'}{dy'}$, and $\frac{d^2x'}{dy'^2}$ in the general expression for the radius of the osculating circle, and then substituting the above values of A, B, D, and x' in the result, we have

$$R_z = -\frac{\left(1 + \frac{d x'^2}{d y'^2}\right)^{\frac{3}{2}}}{\frac{d^2 x'}{d y'^2}} = \frac{A x' + \frac{1}{2} D}{B} = \frac{a\,(1 - e^2)}{(1 - e^2 + e^2 \cos^2 l \cos^2 z)\,(1 - e^2 \sin^2 l)^{\frac{1}{2}}}$$

$$= \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2 z - e^2 \sin^2 l \cos^2 z)\,(1 - e^2 \sin^2 l)^{\frac{1}{2}}}$$

$$= \frac{a\,(1 - e^2)}{(\sin^2 z + \cos^2 z - e^2 \sin^2 z - e^2 \sin^2 l \cos^2 z)\,(1 - e^2 \sin^2 l)}$$

$$= \frac{a\,(1 - e^2)}{(1 - e^2)\,(1 - e^2 \sin^2 l)^{\frac{1}{2}} \sin^2 z + (1 - e^2 \sin^2 l)^{\frac{3}{2}} \cos^2 z},$$

which is the general expression for the radius of curvature of any normal ellipse on the spheroid making, at any latitude l, any angle z with the meridian.

When the normal ellipse BL coincides with the meridian, the angle $z = 0$, $\sin z = 0$, $\cos z = 1$, and R_z becomes $R_m = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}}$, which is the radius of curvature of the meridian at the latitude l. When the normal ellipse $B L$ is at right angles to the meridian, the angle $z = 90°$, $\sin z = 1$, $\cos z = 0$, and R_z becomes $R_n = \frac{a}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}}$, which is the radius of curvature of the perpendicular to the meridian at the latitude l. If $l = 0$, $\sin l = 0$, and we have $R_m = a\,(1 - e^2)$ and $R_n = a$. If $l = 90°$, $\sin l = 1$, and we have $R_m = R_n = \frac{a}{(1 - e^2)^{\frac{1}{2}}}$. In the former case the normal ellipse $B L$ coincides with the equator, while in the latter it coincides with the meridian at the pole.

The radius of curvature of any normal ellipse on the spheroid, whether coinciding with the plane of the meridian, at right angles, or oblique to it, may also be obtained in the following manner. The radius of curvature of the meridian may be derived, at once, from the equation of the meridional ellipse $y^2 = (1 - e^2)(a^2 - n^2)$, by substituting the values of the first and second differential coefficients in the general expression for the radius of the osculating circle, and making the proper substitutions and eliminations. It will be found that

$$\frac{d y^2}{d x^2} = -\frac{(1 - e^2)\,x}{a^2 - x^2}, \qquad \text{and} \qquad \frac{d^2 y}{d x^2} = -\frac{a^2 (1 - e^2)^{\frac{1}{2}}}{(a^2 - x^2)^{\frac{3}{2}}},$$

whence $$R_m = \frac{(a^2 - e^2 x^2)^{\frac{3}{2}}}{a^2 (1 - e^2)} = \frac{a (1 - e^2)}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}}.$$

The radius of curvature of the perpendicular to the meridian is the normal LN produced to the axis of the spheroid, whence it is evident that $MN : CM :: LN : R_n$, or $S : x :: n : R_n$ and consequently

$$R_n = \frac{n x}{S} = \frac{n x}{(1 - e^2) x} = \frac{n}{1 - e^2} = \frac{a}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}}.$$

It can then be easily proved that

$$R_z = \frac{R_m R_n}{R_m \sin^2 z + R_n \cos^2 z} {}^{*} = \frac{a (1 - e^2)}{(1 - e^2)(1 - e^2 \sin^2 l)^{\frac{1}{2}} \sin^2 z + (1 - e^2 \sin^2 l)^{\frac{3}{2}} \cos^2 z}.$$

In order to find the length of a small arc δ° of a normal ellipse, in any position, on the surface of the spheroid, we have merely to multiply the radius of curvature at l, the middle latitude of δ°, by π, the ratio of the semi-circumference to the radius of a circle, and divide the product by $\frac{180^\circ}{\delta^\circ}$. If $\delta = 1^\circ$, we have

$$D_n = \frac{\pi}{180} R_n = \frac{\pi}{180} \cdot \frac{a}{(1 - e^2 \sin^2 l)^{\frac{1}{2}}} = \frac{\pi a}{180} (1 - e^2 \sin^2 l)^{-\frac{1}{2}},$$

which is the length of an arc of one degree of the perpendicular to the meridian at the mean latitude l;

$$D_m = \frac{\pi}{180} R_m = \frac{\pi}{180} \cdot \frac{a (1 - e^2)}{(1 - e^2 \sin^2 l)^{\frac{3}{2}}} = \frac{\pi a (1 - e^2)}{180} (1 - e^2 \sin^2 l)^{-\frac{3}{2}},$$

which is the length of an arc of one degree of the meridian at the mean latitude l; and

$$D_z = \frac{\pi}{180} R_z = \frac{\frac{\pi}{180} R_m R_n}{R_m \sin^2 z + R_n \cos^2 z},$$

which is the length of an arc of one degree making any angle z with the meridian, at the mean latitude l.

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* This Formula is wrong in Capt. T. J. LEE's Tables and Formulæ, Article XI. p. 34.

If, in the last expressions for the length of a degree of the perpendicular and meridian, we develop the second factors by the binomial theorem, we have

$$D_n = \frac{\pi a}{180}\left(1 + \tfrac{1}{2} e^2 \sin^2 l + \tfrac{3}{8} e^4 \sin^4 l + \tfrac{5}{16} e^6 \sin^6 l + \tfrac{35}{128} e^8 \sin^8 l + \&c.\right),$$

and

$$D_m = \frac{\pi a\,(1 - e^2)}{180}\left(1 + \tfrac{3}{2} e^2 \sin^2 l + \tfrac{15}{8} e^4 \sin^4 l + \tfrac{35}{16} e^6 \sin^6 l + \tfrac{315}{128} e^8 \sin^8 l + \&c.\right).$$

Now $\pi = 3.141592654$, and, according to the data adopted by the Superintendent of the United States Coast Survey, $a = 6377397.156$ metres, and $e = 0.081696830$. Substituting these values for π, a, and e, in the developed expressions for the length of a degree of the perpendicular and meridian, we have, in powers of $\sin l$,

$$(1)\quad D_n = 111306.57808 + 371.45076 \sin^2 l + 1.85940 \sin^4 l + 0.01034 \sin^6 l + 0.0006 \sin^8 l.$$

$$(2)\quad D_m = 110563.67657 + 1106.91467 \sin^2 l + 9.23495 \sin^4 l + 0.07191 \sin^6 l + 0.00054 \sin^8 l.$$

But $\sin^2 l = 1 - \cos^2 l$; $\sin^4 l = 1 - 2\cos^2 l + \cos^4 l$;
$\sin^6 l = 1 - 3\cos^2 l + 3\cos^4 l - \cos^6 l$;
and $\sin^8 l = 1 - 4\cos^2 l + 6\cos^4 l - 4\cos^6 l + \cos^8 l$.

Substituting these values for the different powers of $\sin l$ in (1) and (2), we have, for the length of a degree of the perpendicular and meridian in powers of $\cos l$,

$$(3)\quad D_n = 111679.89864 - 375.20082 \cos^2 l + 1.89079 \cos^4 l - 0.01058 \cos^6 l + 0.00006 \cos^8 l.$$

$$(4)\quad D_m = 111679.89864 - 1125.60246 \cos^2 l + 9.45392 \cos^4 l - 0.07407 \cos^6 l + 0.00054 \cos^8 l.$$

Now $\sin^2 l = \frac{1}{2} - \frac{1}{2}\cos 2l$; $\sin^4 l = \frac{3}{8} - \frac{1}{2}\cos 2l + \frac{1}{8}\cos 4l$;
$\sin^6 l = \frac{5}{16} - \frac{15}{32}\cos 2l + \frac{3}{16}\cos 4l - \frac{1}{32}\cos 6l$;

and $\sin^8 l = \frac{35}{128} - \frac{7}{16} \cos 2\, l + \frac{7}{32} \cos 4\, l - \frac{1}{16} \cos 6\, l + \frac{1}{128} \cos 8\, l.$ Substituting these values of the different powers of $\sin l$ in (1) and (2), or changing all the negative into positive signs, which changes the values to those of the powers of $\cos l$ instead of $\sin l$, and substituting in (3) and (4), we have, for the length of a degree of the perpendicular and meridian in multiples of $\cos l$,

(5) $D_n = 111493.00398 - 186.65995 \cos 2\, l + 0.23438 \cos 4\, l - 0.00033 \cos 6\, l.$

(6) $D_m = 111120.61963 - 558.10875 \cos 2\, l + 1.16797 \cos 4\, l - 0.00228 \cos 6\, l.$*

If a perpendicular be let fall from the point L in the meridian PQ, upon the semi-axis CP, it will be the radius of the parallel of latitude at that point, and will obviously be equal to the abscissa CM. Hence we have $R_p = R_n \cos l$, the radius of the parallel of latitude, and consequently $D_p = D_n \cos l$, the length of a degree of the parallel at the latitude l. Therefore, multiplying (3) by $\cos l$, we have, for the length of a degree of the parallel at the latitude l in powers of $\cos l$,

(7) $D_p = 111675.89864 \cos l - 375.20082 \cos^3 l + 1.89079 \cos^5 l - 0.01058 \cos^7 l.$

But $\cos^3 l = \frac{3}{4} \cos l + \frac{1}{4} \cos 3\, l$; $\cos^5 l = \frac{5}{8} \cos l + \frac{5}{16} \cos 3\, l + \frac{1}{16} \cos 5\, l$; and $\cos^7 l = \frac{35}{64} \cos l + \frac{21}{64} \cos 3\, l + \frac{7}{64} \cos 5\, l + \frac{1}{64} \cos 7\, l.$ Substituting these values for the different powers of $\cos l$ in (7), we have, for the length of a degree of the parallel at the latitude l in multiples of $\cos l$,

(8) $D_p = 111399.67397 \cos l - 93.21281 \cos 3\, l + 0.11702 \cos 5\, l - 0.00017 \cos 7 l.$

In practice it will be necessary to retain only three decimal figures in these formulas, and it will be found more convenient to use the powers when we work by the logarithmic, and the multiples when we work by the natural cosines. The following are the formulas for

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* The decimal part of the coefficient of $\cos 2\, l$ in this formula is wrong in the Report of the Superintendent of the United States Coast Survey, for the year 1853, p. 100.*

the powers and the multiples in French metres, American yards, and English yards.

I. In Powers of cos l.

$$D_m = \begin{cases} 111679.899 - 1125.602\cos^2 l + 9.454\cos^4 l - 0.074\cos^6 l, \text{ in F. metres.} \\ 122129.741 - 1230.925\cos^2 l + 10.339\cos^4 l - 0.081\cos^6 l, \text{ in A. yards.} \\ 122136.829 - 1230.996\cos^2 l + 10.339\cos^4 l - 0.081\cos^6 l, \text{ in E. yards.} \end{cases}$$

$$D_n = \begin{cases} 111679.899 - 375.201\cos^2 l + 1.891\cos^4 l - 0.011\cos^6 l, \text{ in F. metres.} \\ 122129.741 - 410.308\cos^2 l + 2.068\cos^4 l - 0.012\cos^6 l, \text{ in A. yards.} \\ 122136.829 - 410.332\cos^2 l + 2.068\cos^4 l - 0.012\cos^6 l, \text{ in E. yards.} \end{cases}$$

$$D_p = \begin{cases} 111679.899\cos l - 375.201\cos^3 l + 1.891\cos^5 l - 0.011\cos^7 l, \text{ in F. metres.} \\ 122129.741\cos l - 410.308\cos^3 l + 2.068\cos^5 l - 0.012\cos^7 l, \text{ in A. yards.} \\ 122136.829\cos l - 410.332\cos^3 l + 2.068\cos^5 l - 0.012\cos^7 l, \text{ in E. yards.} \end{cases}$$

II. In Multiples of cos l.

$$D_m = \begin{cases} 111120.620 - 558.109\cos 2l + 1.168\cos 4l - 0.002\cos 6l, \text{ in F. metr.} \\ 121518.130 - 610.331\cos 2l + 1.277\cos 4l - 0.002\cos 6l, \text{ in A. yds.} \\ 121525.183 - 610.366\cos 2l + 1.277\cos 4l - 0.002\cos 6l, \text{ in E. yds.} \end{cases}$$

$$D_n = \begin{cases} 111493.004 - 186.660\cos 2l + 0.234\cos 4l, \text{ in French metres.} \\ 121925.359 - 204.126\cos 2l + 0.256\cos 4l, \text{ in American yards.} \\ 121932.434 - 204.137\cos 2l + 0.256\cos 4l, \text{ in English yards.} \end{cases}$$

$$D_p = \begin{cases} 111399.674\cos l - 93.213\cos 3l + 0.117\cos 5l, \text{ in French metres.} \\ 121823.296\cos l - 101.935\cos 3l + 0.128\cos 5l, \text{ in American yards.} \\ 121830.366\cos l - 101.941\cos 3l + 0.128\cos 5l, \text{ in English yards.} \end{cases}$$

ON SPHERICAL ANALYSIS.

By George Eastwood, Saxonville, Mass.

[Continued from Page 304, Vol. II.]

Proposition VI.

To find the intersection of two given great circles of the sphere.

In this proposition it is manifest that the abscissa, X, and the ordinate, Y, of the point of intersection are common to both the given circles. If therefore we define the equation of one of them by

$y = \tau x + \beta$, the equation of the other by $y = \tau' x + \beta'$, and the equation of their common point of intersection by

$$Y = \tau X + \beta = \tau' X + \beta',$$

we shall have
$$X = \frac{\beta' - \beta}{\tau - \tau'},$$

and
$$Y = \tau X + \beta = \frac{\tau \beta' - \tau \beta}{\tau - \tau'} + \beta = \frac{\tau \beta' - \tau' \beta}{\tau - \tau'},$$

for the point $X\,Y$ of intersection.

Proposition VII.

To find the intersection of a great with a less circle of the sphere.

Let the centre of the less circle be taken for the origin of co-ordinates; and let, as in Cor. 2 of Prop. II., its equation be represented by

$$y^2 + 2\,x y \cos\omega + x^2 = r^2.$$

If we solve this equation for y, we shall find

$$y = (r^2 - x^2 \sin^2\omega)^{\frac{1}{2}} - x \cos\omega.$$

But by (19)
$$y = \tau x + \beta,$$

$$\therefore\ r^2 - x^2 \sin^2\omega = (\tau x + x \cos\omega + \beta)^2,$$

$$\therefore\ r^2 - \beta^2 = (\tau^2 + 2\,\tau \cos\omega + 1)\,x^2 + 2\,\beta\,(\tau + \cos\omega)\,x,$$

$$\therefore\ x = \frac{(r^2\,(\tau^2 + 2\,\tau\cos\omega + 1) - \beta^2 \sin^2\omega)^{\frac{1}{2}} - \beta\,(\tau + \cos\omega)}{\tau^2 + 2\,\tau\cos\omega + 1} = X,$$

$$\therefore\ y = \tau x + \beta = \frac{\tau\,(r^2\,(\tau^2 + 2\,\tau\cos\omega + 1) - \beta^2\sin^2\omega)^{\frac{1}{2}} + \beta\,(\tau\cos\omega + 1)}{\tau^2 + 2\,\tau\cos\omega + 1)} = Y.$$

Now, that this value of y may be possible, it is necessary that we have

$$r^2 > \frac{\beta^2 \sin^2\omega}{\tau^2 + 2\,\tau\cos\omega + 1};$$

and that the two circles may touch each other, or be tangential, it is necessary and sufficient that

$$(27) \qquad r = \pm\left(\frac{\beta^2\sin^2\omega}{\tau^2 + 2\,\tau\cos\omega + 1}\right)^{\frac{1}{2}} = \frac{\pm\,\beta\sin\omega}{(\tau^2 + 2\,\tau\cos\omega + 1)^{\frac{1}{2}}}.$$

As r, in this case, is equal to the distance between the pole of the less circle and the circumference of the great circle, it follows that (27) makes known by its tangent the distance from the origin of a great circle whose intercepts on the axes of reference are α and β.

If the axes of reference are rectangular, the condition of contact will become $\beta = \pm r(1+\tau^2)^{\frac{1}{2}}$; so that the equation of a great circle of the sphere tangent to a less circle may be written

$$(28) \qquad y = \tau x \pm r(1+\tau^2)^{\frac{1}{2}}.$$

Equation (28) may be applied to the solution of the following problem: —

A less circle of the sphere has its centre at the right angle of a right spherical triangle; it is required to draw a great circle tangent to it, so that the tangents of the intercepted segments of the sides of the triangle may have a given ratio.

There are evidently two solutions to this problem: for if we seek the intersection of the two circles $y = \tau x \pm r(1+\tau^2)^{\frac{1}{2}}$, we see at once that x and y cannot be finite for the two equations

$$y = \tau x + r(1+\tau^2)^{\frac{1}{2}}, \qquad y = \tau x - r(1+\tau^2)^{\frac{1}{2}},$$

when they subsist together. That they may subsist, we must obviously have $x = \infty$; that is, the intersection must be 90° from the pole of the less circle. Hence, in this case, we must have recourse to the latitude of the point of intersection.

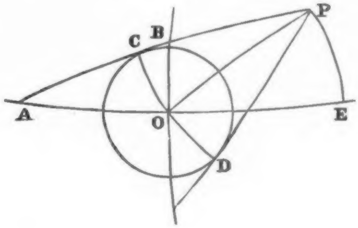
Take $\bar{y}$ for the latitude of the point of intersection, then by (5) $y = \frac{\bar{y}}{\cos x}$; so that the equations of the two tangent circles may be written

$$\bar{y} = \tau \sin x + r(1+\tau^2)^{\frac{1}{2}} \cos x, \qquad \bar{y} = \tau \sin x - r(1+\tau^2)^{\frac{1}{2}} \cos x,$$

from which we readily deduce $\cos x = 0$, or $x = \frac{\pi}{2}$, and $\therefore \bar{y} = \tau$.

As a further illustration of the subject, let P, in the annexed

diagram, be the point of intersection of the two tangent circles, C and D their points of contact with the given circle, and $A\,O$, $O\,B$ the intercepted segments of the sides of the given triangle. Then, because $O\,P = O\,E$ is a quadrantal arc of a great circle, and the angle



$$P\,O\,C = \angle\,P\,O\,D = \frac{\pi}{2},$$

we have the latitude

$$P\,E = \angle\,P\,O\,E = \frac{\pi}{2} - \angle\,D\,O\,E = \frac{\pi}{2} - \angle\,A\,O\,C.$$

And, because the angles $A\,O\,B$, $A\,C\,O$, $B\,C\,O$ are each right angles, the spherical right triangles $A\,C\,O$, $B\,C\,O$ give

$$\tan CO = \tan AO \,.\, \cos AOC = \tan BO \,.\, \cos BOC = \tan BO \,.\, \sin AOC;$$

$$\therefore \cot AOC = \frac{\tan B\,O}{\tan A\,O} = \frac{\beta}{-\alpha} = \tau = \tan P\,E.$$

If $x'\,y'$ be the co-ordinates of the point of contact C or D, the equation of the tangent $A\,P$ or $P\,D$ may be written

$$(29) \qquad y\,y' + x\,x' = r^2.$$

For, by (28), the equation of the point of contact is

$$y' = \tau\,x' + r^2\,(1 + \tau^2)^{\frac{1}{2}},$$

in which $r = (x^2 + y^2)^{\frac{1}{2}}$.

Resolving therefore for τ, we find

$$\tau = -\frac{x'}{y'} = \frac{-1}{\frac{y'}{x'}} = -\frac{\beta}{\alpha},$$

the substitution of which in (28) gives $y\,y' + x\,x' = r^2$.

Proposition VIII.

To find the distance between two given points on the surface of the sphere.

Designating the latitudes of the given points by $\bar{y}_1$, $\bar{y}_2$, the longi-

tudes by x', x'', and their distance apart by $\varDelta$, we shall have from the fundamental spherical triangle,

$$\cos \varDelta = \sin \bar{y}_1 \sin \bar{y}_2 + \cos \bar{y}_1 \cos \bar{y}_2 \cos (x' - x''),$$
$$= \frac{\bar{y}_1 \bar{y}_2 + \cos x' \cos x'' + \sin x' \sin x''}{(1+\bar{y}^2)^{\frac{1}{2}} (1+\bar{y}^2)^{\frac{1}{2}}};$$

in which $\sin \bar{y}_1 \sin \bar{y}_2$ and $\cos \bar{y}_1 \cos \bar{y}_2$ are replaced by their conventional equivalents $\bar{y}_1 \bar{y}_2$ and $(1+\bar{y}_1^2)^{\frac{1}{2}} (1+\bar{y}_2^2)^{\frac{1}{2}}$.

If now we introduce, for the geographical co-ordinates $\bar{y}_1$ and $\bar{y}_2$, the geometrical co-ordinates $y' \cos x'$ and $y'' \cos x''$, or their equals, $\frac{y'}{(1-x'^2)^{\frac{1}{2}}}$ and $\frac{y''}{(1-x'')^{\frac{1}{2}}}$, we shall have

$$(30) \qquad \cos \varDelta = \frac{y' y'' + x' x'' + 1}{\pm (1 + x'^2 + y'^2)^{\frac{1}{2}} (1 + x''^2 + y''^2)^{\frac{1}{2}}},$$

$$(31) \qquad \sin \varDelta = \frac{\{(x'' - x')^2 + (y'' - y')^2 + (x' y'' - x'' y')^2\}^{\frac{1}{2}}}{(1 + x'^2 + y'^2)^{\frac{1}{2}} (1 - x''^2 + y''^2)^{\frac{1}{2}}},$$

$$(32) \qquad \tan \varDelta = \frac{\pm \{(x'' - x')^2 + (y'' - y')^2 + (x' y'' - x'' y')^2\}^{\frac{1}{2}}}{y' y'' + x' x'' 1}.$$

In these formulas the double signs indicate the distance of the point $x' y'$ from the point diametrically opposite to the point $x'' y''$, as well as from the point $x'' y''$ itself. This will be obvious when it is remembered that two points on the surface of the sphere may be joined by two different arcs of great circles, which, taken together, make up the whole circumference of a great circle.

If in equation (32) we regard $\varDelta$ as constant, and x', y', as variables, we shall have the equation of a less circle of the sphere referred to an external origin, in functions of the co-ordinates x'', y'', of its pole, and of its radius $\varDelta$.

When two points on the surface of the spnere are at a distance of 90° from each other, the cosine of this distance is zero; we shall therefore have, between their co-ordinates x', y', x'', y'', the relation

$$(33) \qquad y' y'' + x' x'' + 1 = 0.$$

When two great circles intersect each other at right angles, their poles will be 90° apart; hence the relation (33) will hold for the co-ordinates of their poles.

If two great circles are represented by the equations

$$Mx + Ny = P, \qquad M'x + N'y = P',$$

the relation (33) will give

$$MM' + NN' + PP' = 0. \tag{34}$$

And, generally, if two great circles be defined by the equations

$$\frac{y}{\beta} + \frac{x}{\alpha} = 1, \qquad \frac{y}{\beta'} + \frac{x}{\alpha'} = 1\,;$$

the relation which expresses that they cut each other at right angles will be truly defined by

$$\frac{1}{\beta'\beta} + \frac{1}{\alpha'\alpha} + 1 = 0. \tag{35}$$

PROPOSITION IX.

To find an expression for the angle formed by the intersection of two great circles of the sphere.

The angle formed by the intersection of the two circles is evidently equal to the distance between their poles; ∴ if θ represents this distance, and if $x', y', x'' y''$, be the co-ordinates of the poles, equation (30) will give

$$\cos\theta = \frac{y'y'' + x'x'' + 1}{(1 + x'^2 + y'^2)^{\frac{1}{2}}\,(1 + x'' + y''^2)^{\frac{1}{2}}},$$

for the cosine of the required angle.

But if the circles be represented by $\alpha y + \beta x = \alpha\beta$ and $\alpha' y + \beta' x = \alpha'\beta'$, Prop. IV., gives

$$x' = -\frac{1}{\alpha}, \quad y' = -\frac{1}{\beta} \quad \text{and} \quad x'' = -\frac{1}{\alpha'}, \quad y'' = -\frac{1}{\beta'}.$$

Hence

$$\cos\theta = \frac{\alpha\alpha' + \beta\beta' + \alpha\alpha'\beta\beta'}{(\alpha^2 + \beta^2 + \alpha^2\beta^2)^{\frac{1}{2}}\,(\alpha'^2 + \beta'^2 + \alpha'^2\beta'^2)^{\frac{1}{2}}}. \tag{36}$$

Equation (36) shows that if

$$(37) \qquad \alpha\,\alpha' + \beta\,\beta' + \alpha\,\alpha'\,\beta\,\beta' = 0,$$

the two circles will be perpendicular to each other.

PROPOSITION X.

To find the distance of a given point on the surface of the sphere from the circumference of a great circle.

Let the given point be designated by $x'\,y'$, the pole of the given circle by $x''\,y''$, the distance between them by $\varDelta$, and the required distance by $\varDelta'$; then, by (32),

$$\tan \varDelta = \frac{\pm\{(x''-x')^2+(y''-y')^2+(x'\,y''-x''\,y')^2\}^{\frac{1}{2}}}{y'\,y''+x'\,x''+1}.$$

But $\tan \varDelta' = \cot \varDelta$, hence

$$(38) \qquad \tan \varDelta' = \frac{y'\,y''+x'\,x''-1}{\pm\{(x''-x')^2+(y''-y')^2+(x'\,y''-x''\,y')^2\}^{\frac{1}{2}}},$$

makes known the required distance.

If for x'' and y'' we substitute their values, $-\frac{1}{\alpha'}$ and $-\frac{1}{\beta'}$, then (38) becomes

$$(39) \qquad \tan \varDelta' = \frac{-\alpha'\,y'-\beta'\,x'+\alpha'\,\beta'}{\pm\{(\alpha'\,x'-1)^2+(\beta'\,y'-1)^2+(\alpha'\,x'-\beta'\,y')^2\}^{\frac{1}{2}}},$$

Again, since

$$\sin \varDelta = \frac{\tan \varDelta'}{(1+\tan^2 \varDelta')^{\frac{1}{2}}} = \frac{-\alpha'\,y'-\beta'\,x'+\alpha'\,\beta'}{\pm\{(\alpha'^2+\beta'^2+\alpha'^2\,\beta'^2)\,(1+x'^2+y'^2)\}^{\frac{1}{2}}},$$

∴ in general, if the equation of a great circle be represented by $\mu\,y+\nu\,x+\varrho=0$, then a perpendicular upon its circumference from a given point, $x'\,y'$, may be determined from the equation

$$(40) \qquad \sin \varDelta = \frac{\mu\,y+\nu\,x+\varrho}{\mp\{(\mu^2+\nu^2+\varrho^2)\,(1+x'^2+y'^2)\}^{\frac{1}{2}}}.$$

PROPOSITION XI.

To find the equation of a great circle passing through a given point, and perpendicular to another given great circle.

Let the given great circle be defined by the equation

$$(a) \qquad \frac{y'}{\beta'} + \frac{x'}{\alpha'} - 1 = 0,$$

and the required one by the equation

$$(b) \qquad \frac{y}{\beta} + \frac{x}{\alpha} - 1 = 0.$$

Then, that the conditions of the problem may be fulfilled, we must have, by (35) and (23),

$$(c) \quad \frac{1}{\alpha\alpha'} + \frac{1}{\beta\beta'} + 1 = 0, \quad y - y' = \tau'(x - x') = -\frac{\beta'}{\alpha'}(x - x').$$

Now, from (*b*) we find

$$(d) \qquad \alpha = \frac{\beta x}{\beta - y},$$

and from (*a*) we find $\qquad \alpha' = \dfrac{\beta' x'}{\beta' - y'};$

$$(e) \qquad \therefore \alpha\alpha' = \frac{\beta\beta' x x'}{(\beta - y)(\beta' - y')}.$$

Hence (*c*) becomes $\quad \dfrac{(\beta - y)(\beta' - y')}{\beta\beta' x x'} + \dfrac{1}{\beta\beta'} + 1 = 0,$

which gives $\qquad \beta' = \dfrac{(\beta - y)y' - x'x}{\beta - y + \beta x'x} = \dfrac{\beta y' - \alpha x'}{(1 + \alpha x')\beta},$

by virtue of (*d*).

Again, from (*e*) we have

$$\alpha' = \frac{\beta\beta' x x'}{(\beta - y)(\beta' - y')\alpha} = \frac{\{(\beta - y)y' - x'x\}\beta}{-(\beta - y)(1 + \beta y')\alpha} = \frac{\beta y' - \alpha x'}{-(1 + \beta y')\alpha}.$$

Hence $\tau' = -\dfrac{\beta'}{\alpha'} = \dfrac{\alpha(1 + \beta y')}{\beta(1 + \alpha x')}$, and therefore the required equation is

$$(41) \qquad y - y' = \frac{\alpha(1 + \beta y')}{\beta(1 + \alpha x')}(x - x').$$

If the co-ordinates of the pole of the given circle be x'' and y'',

then, since $\alpha = -\frac{1}{x''}$ and $\beta = -\frac{1}{y''}$, equation (41) gives

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'),$$

which accords with (25), as it evidently ought to do.

A TREATISE ON DETERMINANTS.

CHAPTER I.

INTRODUCTORY.

1. If, by the usual methods of elimination, we solve the linear equations $a'x + b'y = m'$, $a''x + b''y = m''$, we get

$$x = \frac{m'b'' - m''b'}{a'b'' - a''b'}, \qquad y = \frac{a'm'' - a''m'}{a'b'' - a''b'};$$

whose numerators and common denominator have each that peculiar "alternate symmetry," whereby either changes merely in sign if we transpose any two letters throughout without moving the indices ; e. g. $(m'b'' - m''b'') = -(b'm'' - b''m')$. So also, if we transpose two indices throughout without moving the letters; e. g. $(m'b'' - m''b') = -(m''b' - m'b'')$. Indeed the symmetry of either polynomial is precisely the same with respect to its indices as with respect to its letters. Moreover, the numerator of x or of y is formed from the common denominator, by writing m instead of a or instead of b.

All these properties belong as well to any n values x, y, &c., determined by n given linear equations. Thus,

$$x = \frac{m'b''c''' + m''b'''c' + m'''b'c'' - m'''b''c' - m''b'c''' - m'b'''c''}{a'b''c''' + a''b'''c' + a'''b'c'' - a'''b''c' - a''b'c''' - a'b'''c''},$$

$$y = \frac{a'm''c''' + a''m'''c' + a'''m'c'' - a'''m''c' - a''m'c''' - a'm'''c''}{a'b''c''' + a''b'''c' + a'''b'c'' - a'''b''c' - a''b'c''' - a'b'''c''},$$

$$z = \frac{a'b''m''' + a''b'''m' + a'''b'm'' - a'''b''m' - a''b'm''' - a'b'''m''}{a'b''c''' + a''b'''c' + a'''b'c'' - a'''b''c' - a''b'c''' - a'b'''c''},$$

which evidently satisfy the equations

$$a'x + b'y + c'z = m', \quad a''x + b''y + c''z = m'', \quad a'''x + b'''y + c'''z = m''',$$

have these properties.

Again, the numerators and denominator of the set of values obtained from n equations may be found, by a simple law, from the values that satisfy sets of $(n-1)$ equations. Thus the above common denominator of x, y, z, may be written

$$a'(b''c''' - b'''c'') + a''(b'''c' - b'c''') + a'''(b'c'' - b''c'),$$

or

$$a'(b''c''' - c''b''') + b'(c''a''' - a''c''') + c'(a''b''' - b''a''');$$

whose parentheses are the denominators of those values that satisfy the equations

$$b'\xi + c'\eta = m', \quad b''\xi + c''\eta = m'', \quad b'''\xi + c'''\eta = m''';$$

or else the equations

$$a''\xi + a'''\eta = A, \quad b''\xi + b'''\eta = B, \quad c''\xi + c'''\eta = C,$$

taken two and two.

When the common denominator vanishes, some or all of $x, y, \ldots$ must become infinite, indicating that the given equations are incompatible; unless all $m', m'', \ldots$ likewise vanish, in which case $x, y, \ldots$ are indeterminate, indicating that some of the given equations are deducible from the rest.

2. These numerators and denominators are called DETERMINANTS. Their symmetry should appear in most investigations where systems of linear equations enter; and their vanishing commonly indicates that certain co-efficients or functions are not independent. Their properties will often aid us when the difficult or characteristic feature of a problem is to eliminate among the equations or to combine the results of elimination.

The subject of Derivatives assumes a linear character, [we may

say, through the systematic omission of the higher infinitesimals,] such that the change of systems of independent variables in differentiating or integrating, the assignment of multiple definite integrals, and the properties of linear differential equations, so important in dynamics, involve determinants.

In Algebra, equations of any degree have been treated as linear equations, whose new variables are powers and products of the old ones, and may be eliminated by determinants. By such elimination among equations derived from a single one by differentiation, we obtain determinants; e. g., the "discriminant" whose vanishing shall express certain equalities among the roots, and which are "invariants." The extensive theory of invariants, &c., has also other connections with determinants.

In Geometry, we naturally work by combining right lines and planes, whose equations are linear. Co-ordinates are often changed by "linear substitution," as in Perspective, and this introduces Invariants, &c. The contents of triangles and pyramids, being definite integrals, are expressed by determinants.

In the Theory of Numbers and elsewhere, it is natural that those determinants whose vanishing changes the character of any problem, should likewise often affect it by their form or smallness when they do *not* vanish. Especially, the discriminant of a quadratic form so *determines* the character of that form, as to have first received from Gauss the name "determinant." (Disq. Arith. § ...)

The preference of linear forms, even in Geometry and by Nature in Dynamics, appears to be not merely because linear equations are the simpler, as quadratics are simpler than cubics; but also because, unlike all higher equations, they give but one value to each variable, which is always real if the co-efficients and absolute terms are so, the degree, too, being never raised by eliminating any of the variables.

3. Though regarded chiefly as a rather simple instrument of investigation, Determinants have also an interest of their own. I shall present what I suppose to be their leading features and their general modes of application.*

What follows will be chiefly from other treatises; most from BRIOSCHI, but much also from SALMON, PEIRCE, FAÀ DE BRUNO, SPOTTISWOODE, and from several memoirs of CAYLEY and others; with occasional suggestions of my own. Writing rather of the theory than of its authors, I could not always mention the latter without more reading, neither do I care to keep their work unchanged.

I shall try to state clearly, in coarse type, what is at first most important; giving more concisely, and in finer print, things which the reader may omit at his option, as the fine print will never be needed to demonstrate anything that follows in coarse. But he would best notice any *examples* in small type which follow and illustrate the large.

Whether for mere *reference*, or to indicate final *results*, formulas, and perhaps verbal statements, will be numbered in one series, thus, (1), (2), ... Some important results may be marked by bold figures, thus, (**2**). Analogous statements may be indicated by accents, as, (2), (2′), (2″). Subscript indices show which members of an equation are referred to; e. g., (2_3) denotes the third member of (2), and (2_{135}) denotes the equality $(2_1) = (2_3) = (2_5)$.

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* For further and instructive examples, see COMBESCURE's French translation of the admirable "Theory of Determinants and their principal Applications," by Prof. BRIOSCHI, of Pavia, 1856, 8vo, pp. 216; for the best elementary exposition of Invariants, &c., see SALMON's Higher Algebra, Dublin, 1859, 8vo, pp. 147; for a condensed and able account of Determinants as applied to Differential Equations, see PEIRCE's Analytic Mechanics, 1855, Chapter X.; † and for the "General Theory of Elimination," see FAÀ DE BRUNO's work, Paris, 1859, pp. 224; SPOTTISWOODE's *Elementary Theorems relating to Determinants*, 1851, 4to, pp. 63.

† Almost the only other American notice of Determinants which I have seen, is an ingenious and suggestive article "On the Resolution of Symmetrical Equations," by Mr. STOCKWELL, *Astronomical Journal*, VI., 145, 1860.

I have given much thought to the determinant-notation, and especially to that of Chapter III. I have sought to select it from existing systems; but, as the best of these do not perfectly harmonize, I have ventured upon some changes, which, if found conducive to symmetry and convenience, will, I hope, be adopted by others.

Notation should be clearly separated into "temporary," adopted but for a single essay or part of one, and requiring redefinition if ever used again; and "permanent," to be used in the same sense in all memoirs by all writers who adopt it, even without redefinition,—yet always changeable *temporarily and by express definition.* But one notation may be permanently used in different senses, if they differ so *widely* or pertain to such different *subjects* as not to interfere. Of course, I have been most careful with such *permanent* notation as I *have not taken from others.*

Further explanations may be given in text or foot-notes where they are needed. Let us now commence our subject again.

[To be Continued.]

Mathematical Monthly Notices.

Concise Mathematical Operations; being a Sequel to the Author's Class-Books. With much additional Matter. By H. N. Robinson, LL.D., formerly Professor of Mathematics in the United States Navy, &c. New York: Ivison and Phinney. 1854.

Elements of Analytical Geometry, and the Differential and Integral Calculus. By H. N. Robinson, LL.D., Author of a Course of Mathematics, &c. New York: Ivison and Phinney, 48 and 50 Walker Street.

We include these two volumes in the same notice, because in the former, the author first presented his peculiar views of the Differential Calculus, about which we are at present mainly concerned. The latter work, as we learn from the publisher's advertisement, is now undergoing revision, and the two subjects which it contains will be published in separate volumes in January, 1861. As this will give the author a good opportunity to make such changes and improvements as the use of the work may have suggested, we beg to call his attention to a few features of the present edition, feeling desirous of doing all in our power to improve the text-

books from which our students get their first ideas of the science. In the Preface to the work on Mathematical Operations, the author says: —

"There has been a great deal of unnecessary controversy about the Differential and Integral Calculus, which we think can and ought to be wiped away. And we have here given a little foretaste of what we shall attempt if circumstances prompt us to write a work on that subject.

"It is not for us to assume that we can make science clearer than others, but we have yet to see the works of an author who has made the least attempt to show the simple elementary nature of this science. They at once commence with the definition of constants and variables, and then direct what to do.

"We have yet to see the first book that expends a word in giving an idea of what the Calculus is, or what is the utility and object of the science, and we charge more than half the obscurity to this fact alone: hence we could not forbear being a little elementary when we came to that subject, and we leave it to those readers, who have formerly studied other works on this science, to say whether we have or can dispel any of the obscurity that has so long hovered around it."

Again, on page 259: —

"One reason for the appearance of this work is that it is required, because able mathematicians have written so obscurely. They seem to have written as I should, were I indifferent whether the reader, or rather the learner, understood me or not."

We propose now to allow the author to state his own case; and then we can judge whether the "unnecessary controversy" has been "wiped away," either in the "little foretaste" where "we could not forbear being a little elementary," or in the work, which "circumstances" prompted.

The assurance with which the subject is approached, and the estimate which is put upon the labors of those who founded the science, would naturally lead the reader of the above passages to anticipate either great success or great failure, according to his knowledge of the subject and his acquaintance with the labors of the great masters of the science. We shall quote the exact language, italics, and punctuation of the author; and first from the former work: —

"The *differential calculus* is the science of minute variations, or of corresponding *small differences* — a science which owes its birth to the varying elements of Astronomy." — p. 302.

One of the author's correspondents asked him why he called a certain explanation an application of the differential calculus, to which he replied as follows: —

"If the operation I sent you is not calculation, I know not what it is — it may therefore be called calculus; and if in any operation small quantities may be omitted on account of their insignificance in relation to larger quantities, the small difference so omitted constitutes the *differential calculus*, and to obtain that general expression, you will see, by looking on page 193 of the Astronomy, that the powers of r above the first were omitted." — p. 314.

Arithmetic may therefore be called calculus; and because we omit decimal quantities, which on account of their smallness do not affect the required accuracy of the result, it is the differential calculus.

"The differential calculus is a branch of Analytical Geometry. It is a science for computing the *ratio* of small differenees." — p. 315.

After giving the differentials of the circular functions, the author says: —

"All can understand how these equations are obtained; but what if we can? says the inquiring student. What use are they? What do we learn by them? It is useless to answer these questions by words only, we must show the answer by the following examples. Because the differential of a logarithm is the differential of the quantity divided by the quantity, therefore $d \log \sin x = \cot x \, dx$. For example, if we assume $x = 25°$, and also assume $dx = 1'$, the differential, or difference between the log sin 25° and the log sin 25° 1' is expressed by $m \cot 25° \times 1'$. We might assume $dx = 2'$ as well as $1'$, without error as far as six places of decimals; but it would not do to assume $dx =$ any large number of minutes; hence the differential calculus must be applied with judgment. Those who are naturally more nice than wise are commonly prejudiced against this science, and such frequently say it is no science at all; however, their objections are of no consequence." — pp. 322, 323.

Is it possible that the author does not know the use of TAYLOR's Theorem?

We shall now give a few extracts from the *Elements of the Calculus*. As a first example the

author takes the equation $y = ax + b$, and, calling h the increment of x, finds $y' - y = ah$. He then says: —

"The first member of this equation is obviously the *increment* of y, whatever be the value of h, and when h is *extremely* small in relation to x, (we will not say *infinitely* small, as that word puzzles, and it is unnecessary) then $y' - y$ is extremely small in relation to y and in that case we write dy in place of $y' - y$, and dx in the place of h." — p. 146.

Here, then, we have the author's theory of the Calculus. About how small he supposes a differential to be we are not informed; at any rate it is unnecessary, it seems, so far as the logic of the subject is concerned, to consider them infinitesimals. Perhaps we are expected to use our judgment here as well as in the applications.

"The terms *fluxion*, *differential*, and *derivative*, all mean the same thing. There are cases, as in *derived polynomials* in algebra, that it would not do to call the derivative a differential, as the increment might be too large." — p. 146.

We might suppose, were it not for the unfortunate reference to derived polynomials in algebra, that this supposed equality of meaning assigned to the derivative was a slip of the pen. But, at any rate, we learn from this quotation that an increment may be "too large."

On pages 149 and 150 we find a reference to the *limiting ratio*, but then it is said to be between the function and its variable. But this simple reference to another theory seems little more than accidental; for when we come to the differentiation of fractions on page 155, the author has got back upon the doctrine of "extremely small (not to say infinitely small)."

Such is our author's theory of the differential calculus; which seems to us strikingly like the calculus of finite differences used as the differential calculus. Indeed, if we were asked, as we have been, to give a definition of our author's differential calculus, we should say that it is a method of finding a change in a function corresponding to a given change in the variable, provided the change in the variable be not too great; in other words, it is a rather coarse way of computing changes in the values of functions using only first differences, and the judgment as to how large we may assume those differences to be.

Next, let us examine some specimens of demonstration, in the light of the theory: —

"Sometimes we have $u = f(y)$, and $y = F(x)$, and require the differential coefficient between the function u and the variable x. For that purpose we first find $\frac{du}{dy}$ from one equation, and then $\frac{dy}{dx}$ from the other, and multiply them together, and we have $\frac{du}{dx}$ as required." — p. 154.

It seems, then, that $\frac{du}{dy}$ and $\frac{dy}{dx}$ are the same as two ordinary fractions, and their product must be reduced to its lowest terms to obtain the required result.

How it follows, that in general, dy, as an arbitrary increment, equals dy as a dependent increment, and especially so, if neither is infinitely small, we are not informed. Indeed it is not a self-evident proposition applicable to functions involving any number of variables, in any theory of the calculus; and the difficulty is simply disguised for the sake of an apparent simplicity, which will deceive none but very young students.

"Hence we observe that whatever value is given to the variable y, *Taylor's theorem*, and the *binomial theorem*, will give the same result. Hence, when one of these theorems *fail* the other *fails*, but we never apply the word fail to the binomial theorem, and it is not clear to us that such an expression should ever be applied to Taylor's theorem."

If it be not clear to the author, how will it be to the student? The remark that "the *failure* is in the hypothesis and not in the theorems," shows that the author was probably correct when he said, "it is not clear to us."

The true question is, Does TAYLOR'S Theorem, for any function whatever, fail to give the true development to any number of terms for a given hypothesis, when some other method would not fail?

On pages 199 – 202 the tangent, sub-tangent, &c. of plane curves are found by means of the Differential Calculus, and on page 202 we find the remark that, "the student will perceive that these results are here obtained far more easily than in analytical geometry, but we are indebted to analytical geometry for the primary equation of the curve." But a note on the bottom of the same page says that, "It is important that the student should observe that this portion of the calculus is pure analytical geometry."

The idea that the calculus is a branch of analytical geometry is also impressed upon the student on pages 209, 217, 220, 229, as well as in the Preface, apparently, without the faintest suspicion that, as a calculus of functions, with its fundamental principles and its logic, it is entirely independent of any of its applications.

"The differential of an increasing quantity is *positive* before arriving at the maximum *zero*, and *negative* afterwards as we perceive by merely inspecting the figure, and *this is a general principle.* In like manner the differential of a decreasing quantity is minus before it attains its minimum point, it is zero at that point, and positive after passing that point. Hence, if the second differential of a function is *minus*, it indicates that the first differential corresponds to a *maximum*, and if *plus* it indicates a *minimum*. The foregoing illustrations are too plain and practical to meet the entire approbation of some minds; therefore, we give the following as more general and abstract."

Why "some minds" should object to correct geometrical illustrations, and what kind of minds they must be if they do, we cannot understand; but we can comprehend why the "more general and abstract" discussion should not meet the entire approbation of some minds; for a cruder one we never examined. Some of the important points are entirely omitted; such, for instance, as the case in which the same values of the variable reduce the second derivative, as well as the first, to zero. Nothing is said of the maxima and minima of functions of more than one variable.

But we must, at present, omit many other points in the volumes before us for want of space. We have given them, and especially the parts devoted to the Differential and Integral Calculus, a careful and thorough examination; and we are constrained to say that we have never had the misfortune to examine two works, so totally unfitted for text-books, both as regards the treatment of the subjects and the style in which they are written.

We shall take an early opportunity of calling attention to other works by the same author.

Objects and Plan of an Institute of Technology; including, I. A Society of Arts; II. A Museum of Arts; and III. A School of Industrial Science. Proposed to be established in Boston. Prepared by direction of the Committee of Associated Institutions of Science and Arts; and addressed to Manufacturers, Merchants, Mechanics, Agriculturists, and other friends of enlightened industry in the Commonwealth.

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Committee.

The Plan contemplates the following Standing Committees. *First.* Committees of Arts. On Mineral Materials; on Organic Materials; on Tools and Instruments; on Machinery and Motive Powers; on Texile Manufactures; on Manufactures of Wood, Leather, Paper, India Rubber, &c.; on Manufactures of Pottery, Glass, and Precious Metals; on Chemical Products and Processes; on Household Economy, including Warming, Illumination, Water-Supply, Ven-

tilation, and the Preparation and Preservation of Food; on Engineering and Architecture; on Commerce, Navigation, and Inland Transport; on the Graphic and Fine Arts. *Second.* Committee on the Museum. *Third.* Committee on the School of Industrial Science. *Fourth.* Committee on Publications. A very well defined outline of the duties of each committee is given; subject to such changes as experience shall hereafter suggest.

The School of Industrial Science and Art, is to be arranged under the following departments: 1st. A School of Design; 2d. A School of Mathematics; 3d. A School of Physics; 4th. A School of Chemistry; 5th. A School of Geology. Such is the general outline of a most admirable plan, with the details so arranged as to form a symmetrical and systematic whole.

The Report begins by setting forth in a clear and forcible manner the need now felt for the establishment of such an Institution, and the practical value of the results which would flow from it. In the following opinion we heartily concur: "In New England, and especially in our own Commonwealth, the time has arrived when, as we believe, the interests of Commerce and the Arts, as well as of General Education, call for the most earnest co-operation of intelligent culture with industrial pursuits. Our success hitherto in the competitions of trade, manufactures, and the other productive Arts, has been the admitted result of the superior intelligence which has inspired our enterprise and guided our activity; but, to secure a steady prosperity in the midst of the busy inventions and rapidly expanding knowledge which mark these pursuits in the leading European nations, we feel that it has become indispensable for us to provide, at least as effectually as they have done, such facilities for practical knowledge, and for the intelligent guidance of enterprise and labor, as may make our progress commensurate, step by step, with the advances of scientific and practical discovery."

We regret that space forbids our giving a full statement of the arrangement of the proposed Museum, and School of Industrial Science and Art. But the whole plan is well matured, and eminently practical; and of the desirableness of establishing such an Institution in Boston, there can be no question. Still, admitting all that its most enthusiastic friends claim for it, can the idea be carried into execution? We believe it can and will be done. It will need wise counsellors, and unyielding friends, who will be satisfied with nothing less than the largest measure of success. But these it has the good fortune to possess in the persons of the Committee under whose direction this comprehensive Report has been prepared. It will also need the most ample means; but these, we are assured by those who know, and have a right to speak in its behalf, will not be wanting; and it only remains to be seen whether our State will add this last and crowning excellence to her unsurpassed educational system by granting the desired location, upon which private munificence will rear the Institution.

On a Chart and Diagram for Facilitating Great-Circle Sailing. By Hugh Godfray, M. A., St. John's College, Cambridge. (From the Transactions of the Cambridge Philosophical Society, Vol. X. Part II.)

On charts, constructed on the *Central or Gnomonic Projection*, great circles are represented by straight lines; and thus, in Great-Circle Sailing, such charts answer the same purpose that Mercator's Charts do in Rhumb Sailing. To see the ship's track it is only necessary to draw a straight line from the ship's place to her destination. The author says: "By the addition of a Course and Distance Diagram I have made this projection answer all the conditions of Great-Circle Sailing with as much, if not more, facility than Mercator's chart does for sailing on a Rhumb. The track is *seen* a straight line, and this being drawn, the various courses and the distances to be run upon each are obtained, as also the distance from the ship to her destination, by a mere inspection of the diagram; and the chart can be used like an ordinary one for pricking off the ship's place from day to day."

Accompanying the Chart and Course and Distance Diagram, we have received from the

author a *Time Azimuth Diagram*, with a short memoir in explanation of its object, use, and construction. By means of the Diagram the true azimuth can be obtained without calculation to within one eighth of a degree. The data required are the latitude, sun's declination and apparent time at ship. It is hardly possible that such simple and elegant graphic solutions of these problems should not soon find their way into general use.

On the Secular Variations and Mutual Relations of the Orbits of the Asteroids. By SIMON NEWCOMB, S. B., A.A.S. (From the Memoirs of the American Academy, New Series, Vol. VIII.) Cambridge: Welch, Bigelow, and Co., Printers to the University. 1860.

The particular object of this Memoir is to submit OLBER'S hypothesis of the origin of the asteroids to as rigorous a test as is consistent with the nature of the problem. If this hypothesis be true, then there must have been, and still be, certain relations among the asteroid orbits; and the test consists in determining whether these relations do in fact exist. The author has arranged the discussion under the following heads: —

§ 1. *Computation of the rigorous expressions in terms of the times of the elements of the asteroids.*

This section is treated with great elegance and skill. All the terms in these expressions, depending upon the elements of the asteroids, are given in tables with the mean distance of the asteroid as an argument; so that other asteroids may, with but trifling labor, be added to the list of twenty-five treated in this memoir, as fast as their elements become sufficiently well known.

§ 2. *Of the possibility that the orbits of all the asteroids once intersected in a common point.*

Carrying back the elements of the asteroid orbits by means of the formulas in § 1, there is found no tendency to a common point of intersection of the orbits.

§ 3. *Have the elements of the asteroid orbits ever been materially affected by a resisting medium?*

Not sensibly.

§ 4. *Of the relations among the mean distances, eccentricities, and inclinations of the orbits of the asteroids; and between their masses and the velocities with which they must have been projected, if* OLBER'S *hypothesis be true.*

The actual relations are not found to be those which would exist if OLBER'S hypothesis be true.

§ 5. *Of certain observed relations among the asteroids which are the necessary or probable result of known causes, and therefore throw no light on the origin of the asteroids.*

This last section is very interesting, and contains satisfactory solutions of several important questions; such as the unequal distribution of the nodes and perihelia of the orbits; the interlinking of certain orbits, &c.

This valuable memoir is by far the most successful discussion ever made to settle one of the most deeply interesting problems in astronomy, which has agitated the scientific world for the last half-century; and will reflect great credit upon the ability of its author.

Plane Trigonometry, crown 8vo, pp. 271; *Spherical Trigonometry*, crown 8vo, pp. 112. For the use of Colleges and Schools. With numerous examples. By I. TODHUNTER, M. A., Fellow and Assistant Tutor of St. John's College, Cambridge. Macmillan and Co.: Cambridge, and 23 Henrietta Street, Covent Garden, London. 1859.

We do not call the attention of teachers and students to these volumes because we think that there are no good text-books upon these subjects in this country. On the contrary, we have a few excellent ones; and we have no hesitation in saying that we know of no foreign work which surpasses PROFESSOR CHAUVENET'S *Treatise on Plane and Spherical Trigonometry* in all the elements of a good text-book, except, perhaps, in its collection of examples for practice. In this respect these volumes are especially worthy the attention of teachers; the one on Plane Trigonometry contains over six hundred examples, and the other about one hundred and fifty. But they are not ordinary works in other respects. We are particularly interested in the

chapters on the Construction of Trigonometrical Tables; Theory of Proportional Parts; Expansions of Trigonometrical Functions; Summation of Trigonometrical Series; Resolution of Trigonometrical Expressions into Factors; Area of a Spherical Triangle. Spherical Excess; Geodetical Operations.

The author, after showing the connection of the values of the old and new Trigonometric Functions, remarks that, "The old definitions give some indications of the origin of the terms *sine*, *cosine*, &c." The word *sine* seems derived from the Latin *sinus*, a bosom, the arc is supposed to represent a bow, and thus gets its name, and the string, half of which represents the sine of half the arc, would come against the breast of the archer. The words *tangent* and *secant* are naturally derived from the old definitions. (See *Penny Cyclopædia*, Art. *Trig.*)

The modern method has now completely superseded the ancient method in English works; it was introduced by DR. PEACOCK. (See PEACOCK'S *Algebra*, Vol. II. p. 157.) It may, however, be observed, that it is stated by PROFESSOR DE MORGAN (*Trig.* and *Double Algebra*, p. 18), that RHETICUS, who gave the first complete trigonometrical table, and invented the secant and cosecant to complete it, used the method of ratios."

We find in the *Spherical Trigonometry*, p. 38, the method of proof of NAPIER'S Rules indicated by NAPIER in his *Mirifici Logarithmorum Canonis Descriptio*, pp. 32, 35.

In this connection the author gives a collection of opinions with respect to the practical utility of NAPIER'S Rules. "In the whole compass of mathematical science there cannot be found, perhaps, rules which more completely attain that which is the proper object of rules, namely, facility and brevity of computation." WOODHOUSE'S *Trig.*, Chap. X. "In the opinion of DELAMBRE (and no one was better qualified by experience to give an opinion) these theorems are best recollected by the practical calculator in their unconnected form." (AIRY'S *Trig.*, *Encyclopædia Metropolitana.*) "There are certain mnemonical formulæ called NAPIER'S *Rules of Circular Parts*, which are generally explained. We do not give them, because we are convinced that they only create confusion instead of assisting the memory."

We had always supposed by dividing a circle into five parts, and writing upon them in order, a, $\frac{\pi}{2}-B$, $\frac{\pi}{2}-c$, $\frac{\pi}{2}-A$, b; then selecting any one of these five quantities, and observing that of the remaining four, two (b, $\frac{\pi}{2}-B$) are adjacent and two ($\frac{\pi}{2}-c$, $\frac{\pi}{2}-A$) opposite; that the rules

$$\begin{aligned}\sin a &= \tan b \tan\left(\frac{\pi}{2}-B\right) = \cos\left(\frac{\pi}{2}-c\right)\cos\left(\frac{\pi}{2}-A\right)\\ &= \tan b \cot B \qquad = \sin c \sin A,\end{aligned}$$

in which the mind associates *sine* and *middle*, *tangent* and *adjacent*, *cosine* and *opposite*, could not be forgotten or misapplied by the student. Such we should think, from the admirable and lucid manner in which the subject is presented, must be the author's opinion.

Our examination of these volumes confirms us in the high opinion we entertain of the author's series of text-books.

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The following lines are quoted from Harper's Magazine for September. They serve to show very truthfully the comparative value of recent and old commendations: —

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